Chern-Simons Framework for Particles and Quantum Gravity

Risto Raitio*
Helsinki Institute of Physics, P.O. Box 64,
00014 University of Helsinki, Finland

August 16, 2024

Abstract

A novel particle-quantum gravity connection is pointed out and reviewed. It is obtained from a combination of bottom-up and top-down approaches originating from Chern-Simons action of supersymmetric fields. The former starts from the author's composite particle model with spontaneously broken Chern-Simons action binding. The latter approach of other authors incorporates massive spinning fields into the Euclidean path integral of three dimensional quantum gravity via a Chern-Simons formulation. Fundamental matter, as defined in this article, and quantum gravity are conjectured to follow from a unique supersymmetric Chern-Simons action.

*E-mail: risto.raitio@helsinki.fi

Contents

1 Introduction

The purpose of this article is to point out a fundamental connection between a particle model of this author and quantum gravity theory of other authors. We review our bottom-up composite particle physics model and a top-down quantum gravity proposal of other authors. Both approaches lead to a unique framework in theory space.

In Part I, we introduce $N = 1$ supersymmetric preon level vector and chiral multiplets together with 3d Chern-Simons action to build a particle model beyond the standard model. In Part II, the case of $N = 2$ extension with massive spinning fields in Euclidian path integral formulation is analyzed with the same Chern-Simons action and supermultiplets as in Part I. A proposal for 3d quantum gravity is elaborated. In nutshell, the standard model is reinforced by two important elements: the dark sector and gravity.

Part I Symmetry and Wave Functions

2 Composite Particles

The setup for our composite particle scenario is as follows:

- Unbroken supersymmetry is adopted for fundamental particles.[1] Dividing standard model (SM) fermions into three preons[2] a binding mechanism has been constructed using spontaneously broken 3d Chern-Simons theory.
- Preons, or chernons, are provided with two unbroken internal gauge symmetries, $U(1)$ for charge and $SU(3)$ for color.
- There must be freedom and predictions for dark matter and dark energy.
- The scenario should match the cosmological standard model with preheating observational data and baryon asymmetry of matter.

Gravitation is introduced in the form 3d Chern-Simons theory for single-particle states of massive spin s fields living on dS_3, with de Sitter radius ℓ_{dS}, as representations of $\mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R$. It turns out that the partition function for Chern-Simons theory connection $A_{L/R}$ can be calculated. Furthermore, the Chern-Simons path integral can be evaluated to any order in G_N perturbation theory.

The above properties make our preon scenario a worthy candidate beyond SM, call it Unbroken Supersymmetric SM (USSM), which includes all four interactions on quantum level. On the other hand, the generation problem as a composite system excitations and many details remain to be calculated or cannot be done since data are not available. Finally, the scenario should, if possible, indicate the direction to a UV finite theory.

3 Extending the Wess-Zumino action

The divisive point of the chernon model for visible and dark matter is the following: supersymmetry should be unbroken and imple-

[1] The Minimal Supersymmetric SM does not fulfill this requirement (it leads rather to quark and lepton kind of "double counting").

[2] Preons, or here chernons, are free particles above the energy scale Λ_{cr}, numerically abou t $\sim 10^{10} - 10^{16}$ GeV. It is close to reheating scale T_R and the grand unified theory (GUT) scale. At Λ_{cr} chernons make a phase transition by an attractive Chern-Simons model interaction into composite states of standard model quarks and leptons, including gauge interactions. Chernons have undergone "second quarkization".

mented so that all particles needed to describe nature are written together with their superpartners like in the Lagrangians ((1) - (3)) of this model. Our method was introduced in [1, 2]. The result turned out to have close resemblance to the Wess-Zumino (WZ) model [3], which contains three neutral fields: a spinor m, the real fields s and p with $J^P = \frac{1}{2}^+, 0^+$, and 0^-, respectively. The kinetic WZ Lagrangian is

$$\mathcal{L}_{\mathrm{WZ}} = -\frac{1}{2}\bar{m}\gamma^\mu\partial_\mu m - \frac{1}{2}(\partial s)^2 - \frac{1}{2}(\partial p)^2 \tag{1}$$

where m and s form the left-hand chiral supermultiplet. We assume that the pseudoscalar p is the axion [4], and denote it below as a. It has a fermionic superparther, the axino n, a candidate for dark matter but not discussed further here.

To include charged matter we define the following charged chiral field Lagrangian for fermion m^-, complex scalar s^- and the electromagnetic field tensor $F_{\mu\nu}$

$$\mathcal{L}_{\mathrm{WZ}_{\mathrm{Charge}}} = -\frac{1}{2}\bar{m}^-\gamma^\mu\partial_\mu m^- - \frac{1}{2}(\partial s^-)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{2}$$

We set color to the neutral fermion $m \rightarrow m_i^0$ $(i = R, G, B)$ in (1). The color sector Lagrangian is then

$$\mathcal{L}_{\mathrm{WZ}_{\mathrm{Color}}} = -\frac{1}{2}\sum_{i=R,G,B}\left[\bar{m}_i^0\gamma^\mu\partial_\mu m_i^0 - \frac{1}{2}(\partial g_i)^2\right] \tag{3}$$

We now have the supermultiplets shown in table 1.

Multiplet	Particle, Sparticle
chiral multiplets spins 0, 1/2	s^-, m^-; a, n
vector multiplets spins 1/2, 1	m^0, γ; m_i, g_i

Table 1: The particle s^- is a neutral scalar particle. The particles m^-, m^0 are charged and neutral, respectively, Dirac spinors. The a is axion and n axino. m^0 is color singlet particle and γ is the photon. m_i and g_i (i = R, G, B) are zero charge color triplet fermions and bosons, respectively.

Note that in table 1 there is a zero charge quark triplet m_i but no gluon octet. Instead, supersymmetry demands the gluons to appear only in triplets at this stage of cosmological evolution. The dark sector we get from (3) and the m_i.

The matter-chernon correspondence for the first two flavors (r = 1, 2; i.e. the first generation) is indicated in table 2 for left handed particles.

SM Matter 1st gen.	Chernon state
ν_e	$m_R^0 m_G^0 m_B^0$
u_R	$m^+ m^+ m_R^0$
u_G	$m^+ m^+ m_G^0$
u_B	$m^+ m^+ m_B^0$
e^-	$m^- m^- m^-$
d_R	$m^- m_G^0 m_B^0$
d_G	$m^- m_B^0 m_R^0$
d_B	$m^- m_R^0 m_G^0$
W-Z Dark Matter	Particle
boson (or BC)	s, axion(s)
e'	axino n
meson, baryon o	$n\bar{n}, 3n$
nuclei (atoms with γ')	multi n
celestial bodies	any dark stuff
black holes	anything (neutral)

Table 2: Visible and Dark Matter with corresponding particles and chernon composites. m_i^0 (i = R, G, B) is color triplet, $m^\pm$ are color singlets of charge $\pm 1/3$. e' and γ' refer to dark electron and dark photon, respectively. BC stands for Bose condensate. Chernons obey anyon statistics.

After quarks have been formed by the process described in section [6] the SM octet of gluons will emerge because it is known that fractional charge states have not been observed in nature. To make observable color neutral, integer charge states (baryons and mesons) possible we proceed as follows. The local $SU(3)_{color}$ octet structure is formed by quark-antiquark composite pairs as follows (with only color charge indicated):

$$\text{Gluons}: \mathrm{R\bar{G}, R\bar{B}, G\bar{R}, G\bar{B}, B\bar{R}, B\bar{G}}, \frac{1}{\sqrt{2}}(\mathrm{R\bar{R} - G\bar{G}}), \frac{1}{\sqrt{6}}(\mathrm{R\bar{R} + G\bar{G} - 2B\bar{B}}) \tag{4}$$

With the gluon triplet the first hunch is that they form, with octet gluons now available, the $3 \otimes 3 \otimes 3 = 10 \oplus 8 \oplus 8 \oplus 1$ bosonic states with spins 1 and 3. These three gluon coupling states would need a separate investigation.

Finally, we introduce the weak interaction. After the SM quarks, gluons and leptons have been formed at scale Λ_{cr} there is no more observable supersymmetry in nature [5]. To avoid a more complicated vector supermultiplet in table 1, we may append the standard model electroweak interaction in our model as a $SU()2)_Y$ Higgs extension with the weak bosons presented as composite pairs like gluons in (4). The standard model has now been heuristically derived.

Standard model and dark matter is formed by chernon composites in the very early universe at temperature about the reheating value T_R. Due to spontaneous symmetry breaking by a heavy Higgs-like particle the Chern-Simons action can provide a binding force stronger than Coulomb repulsion between equal charge chernons. Details of chernon binding and a mechanism for baryon asymmetry in the Universe are presented in [6, 7]. Here we mention the action used

$$S = \frac{k}{4\pi} \int_M \mathrm{tr}(\mathrm{A} \wedge \mathrm{dA} + \frac{2}{3} \mathrm{A} \wedge \mathrm{A} \wedge \mathrm{A}) \tag{5}$$

and the gauge invariant effective potential for chernon scattering obtained in [8, 9]

$$V_{\mathrm{CS}}(r) = \frac{e^2}{2\pi} \left[1 - \frac{\theta}{m_{ch}} \right] K_0(\theta r) + \frac{1}{m_{ch} r^2} \left\{ l - \frac{e^2}{2\pi\theta} [1 - \theta r K_1(\theta r)] \right\}^2 \tag{6}$$

where $K_0(x)$ and $K_1(x)$ are the modified Bessel functions and l is the angular momentum ($l = 0$ in this note). In (6) the first term [ ] corresponds to the electromagnetic potential, the second one $\{ \}^2$ contains the centrifugal barrier $\left(l/mr^2\right)$, the Aharonov-Bohm term and the two photon exchange term.

Part II Subtleties and localization

Part II material is glued to Part I. A good starting point in references is [10]. We try to collect introductory material for phenomenologists interested in quantum gravity.

4 Chern-Simons gravity

Low-dimensional gravity is an exciting arena to explore and test the gravitational path integral. In two and three spacetime dimensions, there is no propagating graviton and all of the effective degrees of freedom are long-range. A prime example of this phenomenon is the rewriting of pure Einstein gravity with a cosmological constant (of either sign) as a Chern-Simons gauge theory [11] which is the quintessential example of a topological field theory in three-dimensions. A full leveraging of this fact allows the exact evaluation of the gravitational path integral either about a saddle point [12] or as a non-perturbative sum over saddles [13]. While Chern-Simons gravity is not a UV-complete model of quantum gravity [13], its all-loop exactness provides strong tests for potential microscopic models in the spirit of e.g. [14].

One feature that is expected of a UV-complete model of quantum gravity is that it includes matter, in particular massive fields that couple to gravity.[3] The manifest topological invariance that makes Chern-Simons theory so powerful as a description of the gravitational path integral also presents a challenge to incorporating matter. On a practical level, this is simple to illustrate: the action of a massive field theory minimally coupled to a geometry involves both inverse metrics and metric determinants. The rewriting of these terms as Chern-Simons connections is highly non-linear and indicates that integrating out of the massive field will result in a non-local effective action. However, we can take inspiration from the general philosophy that the low-energy avatar of the worldline of a massive degree of freedom is a line-operator of the effective gauge theory [15].

This philosophy was made precise in [16] for massive scalar fields minimally coupled to gravity with a positive cosmological constant.

[3] We take a pragmatic attitude to UV-completeness.

The key result was the expression of the one-loop determinant of a massive scalar field coupled to a background metric, $g_{\mu\nu}$, as a gauge invariant object of the Chern-Simons connections, $A_{L/R}$:

$$Z_{\text{scalar}}[g_{\mu\nu}] = \exp \frac{1}{4}\mathbb{W}[A_L, A_R] \ . \tag{7}$$

The object $\mathbb{W}[A_L, A_R]$, coined the Wilson spool, is a collection of Wilson loop operators wrapped many times around cycles of the base geometry. The equality in (7) is expected to apply to three-dimensional gravity of either sign of cosmological constant, and this was explicitly shown for Euclidean black holes in Anti-de Sitter (i.e., Euclidean BTZ) and Euclidean de Sitter (i.e., the three-sphere S^3) in [16,19]. It has also been upheld on $T\bar{T}$ deformations of AdS_3 [17]. The importance of (7) is not only conceptual, it is practical: it was additionally shown in [16] that certain "exact methods" in Chern-Simons theory (such as Abelianisation [18] (not discussed here) and supersymmetric localization [20]) extend to three-dimensional de Sitter (dS_3) Chern-Simons gravity with the Wilson spool inserted into the path integral. This allows a precise and efficient calculation of the quantum gravitational corrections to Z_{scalar} at any order of perturbation theory of Newton's constant, G_N.

The main result, the generalization of (7) for massive spinning fields, is the following. Consider the local path integral,[4] $Z_{\Delta,\mathsf{s}}$, of a spin-s field $\Phi_{\mu_1\mu_2\ldots\mu_s}$ with mass

$$\frac{m^2}{\Lambda} = (\Delta + \mathsf{s} - 2)(\mathsf{s} - \Delta) \ , \tag{8}$$

minimally coupled to a metric geometry, (M_3, g_{M_3}), where M_3 is topologically either Euclidean BTZ or Euclidean dS_3. Then, it is proposed that

$$\log Z_{\Delta,\mathsf{s}}[g_{M_3}] = \frac{1}{4}\mathbb{W}_{j_L,j_R}[A_L, A_R] \ , \tag{9}$$

where

[4] Including any additional Stückelberg fields to fix its invariances and associated ghosts.

$$\mathbb{W}_{j_L,j_R}[A_L,A_R] = \frac{i}{2}\int_{\mathcal{C}} \frac{\mathrm{d}\alpha}{\alpha}\frac{\cos\left(\frac{\alpha}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)}\left(1+\mathsf{s}^2\sin^2\left(\frac{\alpha}{2}\right)\right)\times$$
$$\sum_{\mathsf{R}_L\otimes\mathsf{R}_R} \mathrm{Tr}_{\mathsf{R}_L}\left(\mathcal{P}e^{\frac{\alpha}{2\pi}\oint_\gamma A_L}\right)\mathrm{Tr}_{\mathsf{R}_R}\left(\mathcal{P}e^{-\frac{\alpha}{2\pi}\oint_\gamma A_R}\right) \ . \quad (10)$$

The details of $\mathbb{W}_{j_L,j_R}$ will be made explicit below, however let us briefly summarize the parts appearing in (10). The Chern-Simons connections, $A_{L/R}$, are related to the metric, g_{M_3}, in (9) through the usual Chern-Simons gravity dictionary and they are integrated over a non-trivial cycle, γ, of the base geometry. The representations, $\mathsf{R}_{L/R}$, appearing in the Wilson loops are summed over a set determined by the mass and spin, (Δ,s), of (9) and labeled by weights (j_L,j_R). Lastly the parameter α is integrated along a contour, $\mathcal{C}$, determined by a regularization scheme appropriate for the sign of cosmological constant. The ultimate effect of the α integral is to implement a "winding" of the Wilson loop operators around γ; this occurs through the summing the residues of the poles of its measure (as well as any of representation traces themselves). The above object, (10), is coined the spinning Wilson spool.

5 Spinning spool on S3

The construction of the Wilson spool for massive spin-s fields on S^3 goes as follows. An expression is derived for the one-loop determinant of these fields on S^3 in terms of the representations of $\mathfrak{su}(2)_L\otimes\mathfrak{su}(2)_R$ constructed in the previous section.

To start, let us describe the path integral of a single massive spin-s field, with no self-interactions. The local partition function for this theory contains a symmetric spin-s tensor, $\Phi_{\mu_1\mu_2\ldots\mu_\mathsf{s}}$, as well as a tower of Stückelberg fields which enforce that $\Phi_{\mu_1\mu_2\ldots\mu_\mathsf{s}}$ is transverse and traceless [21]:

$$\nabla^\nu\Phi_{\nu\mu_2\ldots\mu_\mathsf{s}} = \Phi^\nu{}_{\nu\mu_3\ldots\mu_\mathsf{s}} = 0 \ . \quad (11)$$

As emphasized in [14], on a compact manifold the path integral over symmetric, transverse, traceless (STT) tensors leads to non-local divergences which cannot be canceled by local counterterms. This path integral must be compensated by the path integral over the Stückelberg fields and ghosts which leave behind a finite product from integrating over normalizable zero modes. To that end, we write

$$Z_{\Delta,\mathsf{s}} = Z_{\text{zero}} Z_{\text{STT}} \ , \tag{12}$$

where

$$Z_{\text{STT}} = \int [\mathcal{D}\Phi_{\mu_1\mu_2\dots\mu_\mathsf{s}}]_{\text{STT}} e^{-\frac{1}{2}\int \Phi\left(-\nabla^2_{(\mathsf{s})}+\ell^2_{\text{dS}}\bar{m}^2_\mathsf{s}\right)\Phi} \ . \tag{13}$$

Above $\nabla^2_{(\mathsf{s})}$ is the Laplace-Beltrami operator

$$\left[\nabla^2_{(\mathsf{s})}\Phi\right]_{\mu_1\mu_2\dots\mu_\mathsf{s}} = \nabla_\nu\nabla^\nu\Phi_{\mu_1\mu_2\dots\mu_\mathsf{s}} \ , \tag{14}$$

and $\bar{m}^2_\mathsf{s}$ is an effective mass

$$\ell^2_{\text{dS}}\bar{m}^2_\mathsf{s} = \ell^2_{\text{dS}} m^2 + 3\mathsf{s} - \mathsf{s}^2 \ , \tag{15}$$

where we recall that m^2 is the standard mass parameter in dS_3 [14], and related to the representation theory in [16] via

$$\ell^2_{\text{dS}} m^2 = (\Delta + \mathsf{s} - 2)(\mathsf{s} - \Delta) \ . \tag{16}$$

The zero mode contribution in (12) follows from counting conformal Killing tensor modes on S^3 and is given by [14][5]

$$Z_{\text{zero}} = \left[(\Delta-1)(\bar{\Delta}-1)\right]^{\frac{\mathsf{s}^2}{2}} \prod_{n=0}^{\mathsf{s}-2}\left[(\Delta+n)(\bar{\Delta}+n)\right]^{-(n+1+\mathsf{s})(n+1-\mathsf{s})} \ , \tag{17}$$

where we recall that $\bar{\Delta} = 2 - \Delta$.

In the following, we will show that

$$\log Z_{\Delta,\mathsf{s}} = \frac{1}{4}\mathbb{W}_{j_L,j_R} \ , \tag{18}$$

for fields on S^3. That is, we will express $Z_{\Delta,\mathsf{s}}$ as a function of the Chern-Simons connections which we will explicitly construct utilizing the non-standard representation theory of [16]. While the

[5] This is true for $\mathsf{s} \geq 2$ while for $\mathsf{s} = 0, 1$ the product over n is replaced by 1.

construction takes place on a fixed classical background, we will see that $\mathbb{W}_{j_L,j_R}$ is an integral over gauge invariant Wilson loop operators and naturally generalizes into an off-shell operator that can be inserted into the Chern-Simons path integral.

The roadmap to derive (18) is as follows. We will first focus on $Z_{\rm STT}$. This determinant can be evaluated via the method of quasinormal modes pioneered in [22]. We will adapt this method such that each component has a group theoretic interpretation: we will show how defining properties of quasinormal modes can be translated to conditions on the representation of the fields. This follows [16], however, for spinning fields, we will take particular care with the role of global conditions (i.e., Euclidean solutions are regular and single-valued) in isolating physical contributions to the quasinormal mode product. The additional contribution of $Z_{\rm zero}$, which is not part of the quasinormal mode product but crucial for maintaining locality of $Z_{\Delta,\mathsf{s}}$, will permit a Schwinger parameterization of $\log Z_{\Delta,\mathsf{s}}$, regularized by an $i\varepsilon$ prescription. This will organize the quasion-standnormal mode sum into an integral over representation traces of the background holonomies. From this follows the main result.

5.1 A group theory perspective on S3

As the first step in our construction, we will recast the functional determinant

$$Z_{\rm STT} = \det\left(-\nabla^2_{(\mathsf{s})} + \ell^2_{\rm dS}\bar{m}^2_{\mathsf{s}}\right)^{-\frac{1}{2}} \tag{19}$$

in $\mathfrak{su}(2)$ representation theoretic language. We recall that the DHS method [22] instructs us to treat $Z^2_{\rm STT}$ as meromorphic function of Δ. Then, up to a holomorphic function, $Z^2_{\rm STT}$ is equal to the product containing the same zeros and poles. Here $Z_{\rm STT}$ only has poles on states satisfying $(-\nabla^2_{(\mathsf{s})} + \bar{m}^2_{\mathsf{s}})\Phi_{\mu_1\mu_2\cdots\mu_{\mathsf{s}}} = 0$. These are precisely the spin-s quasinormal modes. We have explicitly computed these modes and their product to obtain $Z_{\rm STT}$ directly. Here we will reinterpret these modes in terms of $\mathfrak{su}(2)$ representation theory to obtain an expression natural to the Chern-Simons theory formulation of gravity.

We note that the isometry algebra of the three-sphere is generated by two mutually commuting sets of $\mathfrak{su}(2)$ vector fields $\{\zeta_a\}$ and $\{\bar{\zeta}_b\}$ which are the infinitesimal left and right group actions acting on $S^3 \simeq SU(2)$. On spin-s STT tensors the Casimirs of their Lie derivatives, $\{\mathcal{L}_{\zeta_a}\}$ and $\{\mathcal{L}_{\bar{\zeta}_b}\}$, act as the Laplace-Beltrami operator:

$$-2\delta^{ab}\left(\mathcal{L}_{\zeta_a}\mathcal{L}_{\zeta_b}+\mathcal{L}_{\bar{\zeta}_a}\mathcal{L}_{\bar{\zeta}_b}\right)\Phi_{\mu_1\mu_2\ldots\mu_{\mathsf{s}}}=\left[\nabla^2_{(\mathsf{s})}-\mathsf{s}(\mathsf{s}+1)\right]\Phi_{\mu_1\mu_2\ldots\mu_{\mathsf{s}}}\ . \tag{20}$$

Hence we can write (19) suggestively as

$$Z_{\mathrm{STT}}=\det\left(2c_2^{\mathfrak{su}(2)_L}+2c_2^{\mathfrak{su}(2)_R}+\Delta(2-\Delta)-\mathsf{s}^2\right)^{-\frac{1}{2}}\ . \tag{21}$$

Following the DHS methodology, we then expect Z^2_{STT} to have pole contributions from states in $\mathfrak{su}(2)_L\oplus\mathfrak{su}(2)_R$ representations satisfying

$$\left(-2c_2^{\mathfrak{su}(2)_L}-2c_2^{\mathfrak{su}(2)_R}\right)|\psi\rangle=\left[\Delta(2-\Delta)-\mathsf{s}^2\right]|\psi\rangle\ . \tag{22}$$

This is precisely the condition satisfied by the non-standard representations constructed in [16] with highest weights $(j_L,j_R)=(-\frac{\Delta+\mathsf{s}}{2},-\frac{\Delta-\mathsf{s}}{2})$. We are interested in the poles in Z^2_{STT} arising from weights of the representations $\mathsf{R}_{j_L}\otimes\mathsf{R}_{j_R}$ as we continue Δ in the complex plane.[6] In principle we should consider all representations that satisfy (22), so for a given (Δ,s), we also encounter poles associated to highest weight representations arrived at by sending $\Delta\to\bar{\Delta}=2-\Delta$ as well as $\mathsf{s}\to-\mathsf{s}$.[7] If we define $(j_L,j_R)=(-\frac{\Delta+\mathsf{s}}{2},-\frac{\Delta-\mathsf{s}}{2})$ then we will denote

$$(\bar{j}_L,\bar{j}_R)=\left(-\frac{\bar{\Delta}+\mathsf{s}}{2},-\frac{\bar{\Delta}-\mathsf{s}}{2}\right)\ , \tag{23}$$

[6] It will be important as we progress to take special care of the cases when Δ is such that $j_{L/R}\in\frac{1}{2}\mathbb{N}$; in these cases the weight spaces terminate discontinuously to finite-dimensional representations.

[7] This is consistent with the Lorentzian picture: a spin-s field is built out of $\mathfrak{so}(1,3)$ representations labelled by both $(\Delta,\pm\mathsf{s})$ while the (Δ,s) representation is isomorphic to that labelled by $(\bar{\Delta},-\mathsf{s})$ through the $\mathfrak{so}(1,3)$ shadow map [23].

while $\mathsf{s} \to -\mathsf{s}$ is equivalent to $j_L \leftrightarrow j_R$. We will then have pole contributions from any of the representations appearing in

$$\mathcal{R}_{\Delta,\mathsf{s}} = \left\{ \mathsf{R}_{j_L} \otimes \mathsf{R}_{j_R}, \mathsf{R}_{\bar{j}_L} \otimes \mathsf{R}_{\bar{j}_R}, \mathsf{R}_{j_R} \otimes \mathsf{R}_{j_L}, \mathsf{R}_{\bar{j}_R} \otimes \mathsf{R}_{\bar{j}_L} \right\} \ . \tag{24}$$

We make a special note that for scalar representations ($j_L = j_R = j$) it is sufficient to consider the smaller set

$$\mathcal{R}_{\Delta,\text{scalar}} = \left\{ \mathsf{R}_j \otimes \mathsf{R}_j, \mathsf{R}_{\bar{j}} \otimes \mathsf{R}_{\bar{j}} \right\} \tag{25}$$

as in [16].

The "mass-shell condition" (22) is only a necessary condition to contribute a physical pole to Z^2_{STT}. Functional determinants come with boundary and regularity conditions on their functional domain and we must impose these on weight spaces satisfying (22) to obtain a physical answer. We will state these up-front in an $\mathfrak{su}(2)$ natural language as the following:

Condition I. Single-valued solutions: A configuration constructed from a representation $\mathsf{R}_L \otimes \mathsf{R}_R \in \mathcal{R}$ must return to itself under parallel transport around any closed cycle in the Euclidean manifold.

Condition II. Globally regular solutions: A configuration constructed from a representation $\mathsf{R}_L \otimes \mathsf{R}_R \in \mathcal{R}$ must be globally regular on the Euclidean manifold. When the base space is homogeneous this means $\mathsf{R}_L \otimes \mathsf{R}_R$ lifts from a representation of the isometry algebra to a representation of the isometry group.

Let us first expand upon Condition I for spin-s fields on S^3. A field Φ living in a representation $\mathsf{R}_L \otimes \mathsf{R}_R$ is parallel transported around a cycle, γ, through the background connections:

$$\Phi_f = \mathsf{R}_L \left[\mathcal{P} \exp \left(\oint_\gamma a_L \right) \right] \Phi_i \, \mathsf{R}_R \left[\mathcal{P} \exp \left(- \oint_\gamma a_R \right) \right] \ . \tag{26}$$

When $a_{L/R}$ are flat this conjugation is trivial. However for the backgrounds appropriate for describing the S^3 metric geometry, the background connections take non-trivial holonomies

$$\mathcal{P}\exp\left(\oint_\gamma a_L\right) = u_L^{-1}\, e^{i2\pi L_3 \mathsf{h}_L^{(\gamma)}}\, u_L^{-1}\;, \quad \mathcal{P}\exp\left(\oint_\gamma a_R\right) = u_R^{-1}\, e^{i2\pi \bar{L}_3 \mathsf{h}_R^{(\gamma)}}\, u_R^{-1}\;, \tag{27}$$

when γ wraps one of two lines on the base S^3 [16]. These lines are Hopf linked and Wick rotate to the coordinate positions of the static patch origin and horizon. They yield respective holonomies

$$\gamma_{\text{orig.}}: \quad (\mathsf{h}_L, \mathsf{h}_R) = (1,1)\;, \qquad \gamma_{\text{hor.}}: \quad (\mathsf{h}_L, \mathsf{h}_R) = (1,-1)\;. \tag{28}$$

The salient point is that a single-valued field will satisfy

$$\lambda_L \mathsf{h}_L - \lambda_R \mathsf{h}_R \in \mathbb{Z}\;, \tag{29}$$

for each of the two sets of holonomies in (28) and for all weights (λ_L, λ_R) in the representation $\mathsf{R}_L \otimes \mathsf{R}_R$.

From cycles wrapping the origin weights must satisfy

$$\lambda_L - \lambda_R \in \mathbb{Z} \tag{30}$$

to contribute a pole to Z^2_{STT}. Weights of a highest-weight representation $\mathsf{R}_{j_{L/R}}$ necessarily take the form

$$\lambda_{L/R} = j_{L/R} - p_{L/R}\;, \qquad p_{L/R} \in \mathbb{N}_0\;, \tag{31}$$

simply via the structure of the $\mathfrak{su}(2)$ algebra. Single-valuedness around the static patch origin then requires

$$j_L - j_R \in \mathbb{Z} \qquad \Leftrightarrow \qquad \mathsf{s} \in \mathbb{Z}\;. \tag{32}$$

This condition is the same for all other representations in $\mathcal{R}_{\Delta,\mathsf{s}}$. We pause here to note that while the representation theory only relies on $j_L - j_R \in \mathbb{R}$, we now see that only fields with quantized spin can contribute physical poles to Z^2_{STT}. We will thus fix $\mathsf{s} \in \mathbb{Z}$ and consider the analytic structure of Z^2_{STT} as a function of Δ. This analytic structure is constrained by single-valuedness around the static patch horizon, which requires

$$\lambda_L + \lambda_R \in \mathbb{Z}\;. \tag{33}$$

We will return to this condition shortly.

We now address Condition II, that configurations contributing to Z^2_{STT} are globally regular. Without loss of generality we will state this for $\mathsf{R}_{j_L} \otimes \mathsf{R}_{j_R} \in \mathcal{R}_{\Delta,\mathsf{s}}$. For the S^3 background in question the isometry group acts transitively. Thus regularity at a point guarantees global regularity as long as the isometry group, $SU(2)_L \times SU(2)_R$, acts faithfully on the field in question: that is, $\mathsf{R}_{j_L} \otimes \mathsf{R}_{j_R}$ lifts to a representation of the isometry group. The Peter-Weyl theorem states that these must be finite-dimensional representations of $\mathfrak{su}(2)_L \otimes \mathfrak{su}(2)_R$, where such representations have weights (31) satisfying

$$\lambda_{L/R} = j_{L/R} - p_{L/R} \,, \qquad 0 \leq p_{L/R} \leq 2j_{L/R} \,, \qquad j_{L/R} \in \frac{1}{2}\mathbb{N}_0 \,. \tag{34}$$

To be clear about interpretation: the DHS method instructs us to consider the structure of Z^2_{STT} as Δ continues to the complex plane. The mass-shell condition, (22), then instructs us to consider representations with generically complex highest weights, $j_{L/R} \in \mathbb{C}$. Such representations are non-standard and infinite-dimensional. However Condition II simply tells us that the poles of Z^2_{STT} are located at $\Delta \in \mathbb{Z}_{\leq -\mathsf{s}}$ and the orders of these poles are correctly counted not by weights of an infinite dimensional representation but instead by (34). In this counting we notice that weights of finite dimensional representations of $SU(2)$ are centered about zero and so for any weight satisfying $\lambda_L + \lambda_R = N > 0$ there is a corresponding weight with $\lambda_L + \lambda_R = -N$. Thus for the purposes of counting the number of weights contributing to a particular pole, we can restate (33) as

$$\lambda_L + \lambda_R = |N| \,, \qquad N \in \mathbb{Z} \tag{35}$$

as a necessary condition for incorporating Condition II.

We pause to note that for minimally coupled scalar fields (35) is also sufficient to imply Condition II. Thus the scalar one-loop determinant can be written as

$$Z_{\text{scalar}} = \prod_{\substack{(\lambda_L,\lambda_R) \\ \in \mathsf{R}_j \otimes \mathsf{R}_j}} \prod_{N \in \mathbb{Z}} \left(|N| - \lambda_L - \lambda_R\right)^{-1/2}$$

$$\times \prod_{\substack{(\bar{\lambda}_L,\bar{\lambda}_R) \\ \in \mathsf{R}_{\bar{j}} \otimes \mathsf{R}_{\bar{j}}}} \prod_{\bar{N} \in \mathbb{Z}} \left(|\bar{N}| - \bar{\lambda}_L - \bar{\lambda}_R \right)^{-1/2} , \tag{36}$$

where we have written explicitly the product over the two representations appearing in $\mathcal{R}_{\Delta,\text{scalar}}$. From here the expression of Z_{scalar} as a Wilson spool follows the procedure in [16].

For massive spin-s fields, (35) is no longer sufficient and we must impose additional constraints to reproduce Z^2_{STT}. For a given $|N|$ in (35), Condition II additionally implies the weights $\lambda_{L/R} = j_{L/R} - p_{L/R}$ must satisfy

$$p_L \geq -|N| - \mathsf{s} , \qquad p_R \geq -|N| + \mathsf{s} . \tag{37}$$

While the first of these is always satisfied (for positive s) the second is an additional constraint on counting the order of the poles appearing in Z^2_{STT} and is only non-trivial when $|N| \leq \mathsf{s}$. For these $2\mathsf{s}+1$ cases we observe that $\tilde{p}_R = p_R + |N| - \mathsf{s} \geq 0$ and (35) is equivalently written

$$j_L + j_R - p_L - \tilde{p}_R = \mathsf{s} , \qquad p_L, \tilde{p}_R \geq 0 . \tag{38}$$

We can thus treat this as a condition on the weights $(\lambda_L, \tilde{\lambda}_R) = (j_L - p_L, j_R - \tilde{p}_R)$ of highest-weight representations $\mathsf{R}_{j_L} \otimes \mathsf{R}_{j_R}$ and for each pole arising from this condition being satisfied, it arises $2\mathsf{s}+1$ times.

Applying this same procedure to all representations appearing in $\mathcal{R}_{\Delta,\mathsf{s}}$ we arrive at

$$Z_{\text{STT}} = \prod_{\mathcal{R}_{\Delta,\mathsf{s}}} \prod_{(\lambda_L,\lambda_R)} \left((\mathsf{s} - \lambda_L - \lambda_R)^{-\frac{2\mathsf{s}+1}{2}} \prod_{\substack{N \in \mathbb{Z}, \\ |N| > \mathsf{s}}} (|N| - \lambda_L - \lambda_R)^{-\frac{1}{2}} \right) , \tag{39}$$

where the first product is understood to take the product over all pairs $\mathsf{R}_L \otimes \mathsf{R}_R \in \mathcal{R}_{\Delta,\mathsf{s}}$ and the second product is taken over all weights $(\lambda_L, \lambda_R) \in \mathsf{R}_L \otimes \mathsf{R}_R$ of a particular pair in $\mathcal{R}_{\Delta,\mathsf{s}}$. Where it does not cause confusion we will maintain this shorthand (in both products and sums) for compactness of notation.

As mentioned at the beginning to this section, the local spin-s partition function, $Z_{\Delta,\mathsf{s}}$, includes, in addition to this quasinormal product, the product from integrating over normalizable zero modes, (17):

$$Z_{\Delta,\mathsf{s}} = Z_{\text{zero}} Z_{\text{STT}} \ . \tag{40}$$

In the next section we will show how this combination, with the expression of Z_{STT} as a product over representation weights, (39), will lead to the Wilson spool.

5.2 Constructing the spool

The procedure to cast $\log Z_{\Delta,\mathsf{s}}$ as an integral over Wilson loop operators starts by rearranging (17) and (39). We first make use of the Schwinger parameterization of the logarithm

$$\log M = -\int_{\times}^{\infty} \frac{\mathrm{d}\alpha}{\alpha} e^{-\alpha M} \ , \tag{41}$$

with a regularization of the divergence at $\alpha \to 0$ that we will leave unspecified for now. We will address this regularization through a suitable $i\varepsilon$ prescription below. Applying (41) to (39), we first see that the sum over weights in $\log Z_{\text{STT}}$ can then be organized into representation traces

$$\sum_{(\lambda_L,\lambda_R)} e^{\alpha(\lambda_L+\lambda_R)} = \mathrm{Tr}_{\mathsf{R}_L}\left(e^{\alpha L_3}\right) \mathrm{Tr}_{\mathsf{R}_R}\left(e^{\alpha \bar{L}_3}\right) \ , \tag{42}$$

which are the characters of the non-standard representation.

$$\log(Z_{\text{STT}}) = \frac{1}{2}\int_{\times}^{\infty} \frac{\mathrm{d}\alpha}{\alpha} \left(\sum_{\substack{N\in\mathbb{Z} \\ |N|>\mathsf{s}}} e^{-|N|\alpha} + (2\mathsf{s}+1) e^{-\mathsf{s}\alpha} \right) \times \sum_{\mathcal{R}_{\Delta,\mathsf{s}}} \mathrm{Tr}_{\mathsf{R}_L}\left(e^{\alpha L_3}\right) \mathrm{Tr}_{\mathsf{R}_R}\left(e^{\alpha \bar{L}_3}\right) \ . \tag{43}$$

Similarly, we can introduce a Schwinger parameter to $\log Z_{\text{zero}}$, where now (17) reads

$$\begin{aligned}\log Z_{\text{zero}} = &\int_{\times}^{\infty} \frac{\mathrm{d}\alpha}{\alpha}\left(\sum_{n=0}^{\mathsf{s}-2}((n+1)^2-\mathsf{s}^2)e^{(j_L+j_R-n)\alpha} - \frac{\mathsf{s}^2}{2}e^{(j_L+j_R+1)\alpha}\right)\\ &+ (j_{L/R} \to \bar{j}_{L/R})\\ =& \frac{1}{2}\int_{\times}^{\infty}\frac{\mathrm{d}\alpha}{\alpha}(e^{\alpha}-1)^2 e^{-2\alpha}\left(\sum_{n=0}^{\mathsf{s}-2}((n+1)^2-\mathsf{s}^2)e^{-n\alpha} - \frac{\mathsf{s}^2}{2}e^{\alpha}\right)\\ &\times \sum_{\mathcal{R}_{\Delta,\mathsf{s}}} \mathrm{Tr}_{\mathsf{R}_L}\left(e^{\alpha L_3}\right)\mathrm{Tr}_{\mathsf{R}_R}\left(e^{\alpha \bar{L}_3}\right) . \end{aligned} \tag{44}$$

In the first line, we used $\Delta = -j_L - j_R$, and in the second, the characters to cast this as a trace. The zero mode contribution (44) combines nicely with $\log\left(Z_{\text{STT}}\right)$ to give

$$\log Z_{\Delta,\mathsf{s}} = \frac{1}{2}\int_{\times}^{\infty}\frac{\mathrm{d}\alpha}{\alpha}\left(\frac{\cosh\left(\frac{\alpha}{2}\right)}{\sinh\left(\frac{\alpha}{2}\right)} - \mathsf{s}^2\sinh(\alpha)\right)\sum_{\mathcal{R}_{\Delta,\mathsf{s}}}\mathrm{Tr}_{\mathsf{R}_L}\left(e^{\alpha L_3}\right)\mathrm{Tr}_{\mathsf{R}_R}\left(e^{\alpha \bar{L}_3}\right) , \tag{45}$$

where we used

$$\sum_{n\in\mathbb{Z}} e^{-|n|\alpha} = \frac{\cosh\left(\frac{\alpha}{2}\right)}{\sinh\left(\frac{\alpha}{2}\right)} . \tag{46}$$

At this point we use the even parity of the integrand to regulate the divergence through the following $i\varepsilon$ prescription:

$$\int_{\times}^{\infty}\frac{\mathrm{d}\alpha}{\alpha}f(\alpha) := \lim_{\varepsilon\to 0}\frac{1}{4}\sum_{\pm}\int_{-\infty}^{\infty}\frac{\mathrm{d}\alpha}{\alpha\pm i\varepsilon}f(\alpha\pm i\varepsilon) . \tag{47}$$

This is a choice of regularization scheme for the one-loop determinant. Finally, under a change of integration variables $\alpha \to -i\alpha$ we can write the partition function as

$$\begin{aligned}\log Z_{\Delta,\mathsf{s}} = \frac{i}{8}\int_{\mathcal{C}}\frac{\mathrm{d}\alpha}{\alpha}\left(\frac{\cos\left(\frac{\alpha}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)} + 2\mathsf{s}^2\cos\left(\frac{\alpha}{2}\right)\sin\left(\frac{\alpha}{2}\right)\right)\times\\ \sum_{\mathcal{R}_{\Delta,\mathsf{s}}}\mathrm{Tr}_{\mathsf{R}_L}\left(e^{i\alpha L_3}\right)\mathrm{Tr}_{\mathsf{R}_R}\left(e^{i\alpha \bar{L}_3}\right) ,\end{aligned} \tag{48}$$

where the contour $\mathcal{C}$ runs upwards along the imaginary α axis to the left and right of the divergence at the origin, as depicted in figure 1.

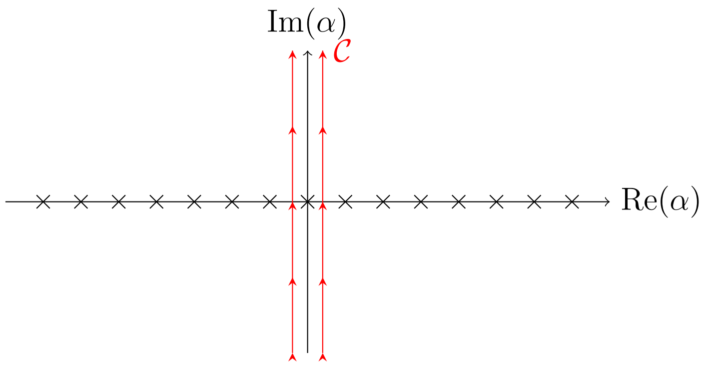

Figure 1: The integration contour regulating the $\alpha \to 0$ divergence.

As a final step we now rewrite the holonomies inside the traces to restore the background connections, arriving at (18) with $\mathbb{W}_{j_L,j_R}$ the spinning Wilson spool:

$$\mathbb{W}_{j_L,j_R}[a_L, a_R] := \frac{i}{2}\int_{\mathcal{C}} \frac{\mathrm{d}\alpha}{\alpha}\frac{\cos\left(\frac{\alpha}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)}\left(1 + 2\mathsf{s}^2\sin^2\left(\frac{\alpha}{2}\right)\right)\times \sum_{\mathcal{R}_{\Delta,\mathsf{s}}} \mathrm{Tr}_{\mathsf{R}_L}\left(\mathcal{P}e^{\frac{\alpha}{2\pi}\oint_\gamma a_L}\right)\mathrm{Tr}_{\mathsf{R}_R}\left(\mathcal{P}e^{-\frac{\alpha}{2\pi}\oint_\gamma a_R}\right) \ , \quad (49)$$

where above $\gamma = \gamma_{\text{hor.}}$ is a cycle wrapping the singular point corresponding to the horizon.

At this point let us make several comments:

- The spinning Wilson spool takes a form similar to that of the scalar spool found in [16,19]; importantly the "operator pieces" of the expression (49) have been organized into gauge invariant Wilson loop operators. The only modification the spinning spool brings is in the integration measure.
- The modification to the integration measure, proportional to s^2, will have the effect of lowering the degree of each pole at $\alpha \in 2\pi\mathbb{Z}$ by two. As we will shortly see, this effect reproduces the "edge partition function" of [14].

- Mathematically the holonomies corresponding to $\gamma_{\text{hor.}}$ appear in the one-loop determinant because they are sensitive to Δ on which Z_{STT} is treated as an meromorphic function. The physics behind this is clear: we are reproducing a one-loop determinant of massive fields. In the worldline quantum mechanics framework this corresponds to averaging over worldlines of a massive particle in the static patch. Such wordlines are timelike and rotate to a contour gauge equivalent to $\gamma_{\text{hor.}}$.

5.3 Testing the Wilson spool

We now uphold equations (18) and (49) by verifying that it indeed reproduces the correct path integral of a massive spinning field on S^3. There are several ways how to evaluate this path integral, and here we will focus on two approaches to use as a comparison. The first is to implement the DHS method traditionally. We evaluate this path integral by explicitly listing the quasinormal modes and applying DHS. The second approach we can compare to are the expressions found in [14], which cast the results in terms of $\mathfrak{so}(1,3)$ characters.

Since we are not turning on gravity ($G_N \to 0$), we evaluate the Wilson loop operators in (49) as characters with the appropriate holonomies in (28). Using the form of our non-standard representation character, we then write

$$\log Z_{\Delta,\mathsf{s}} = -\frac{i}{8}\int_{\mathcal{C}} \frac{\mathrm{d}\alpha}{\alpha}\left(\frac{\cos\left(\frac{\alpha}{2}\right)}{\sin^3\left(\frac{\alpha}{2}\right)} + 2\mathsf{s}^2\frac{\cos\left(\frac{\alpha}{2}\right)}{\sin\left(\frac{\alpha}{2}\right)}\right) e^{i\alpha(1-\Delta)}\ , \tag{50}$$

where we note that the sum over representations in $\mathcal{R}_{\Delta,\mathsf{s}}$ is already neatly packaged into our two contours. We recognize the first term in the parentheses of (50) as twice the on-shell scalar Wilson spool found in [16]. We evaluate both terms by deforming the α contours to run above and below the real α axis to pick up the residues at the poles lying at $2\pi\mathbb{Z}_{\neq 0}$. This deformation is depicted in figure 2.

Summing the towers of poles and expressing the $\mathfrak{su}(2)_{L/R}$ highest-

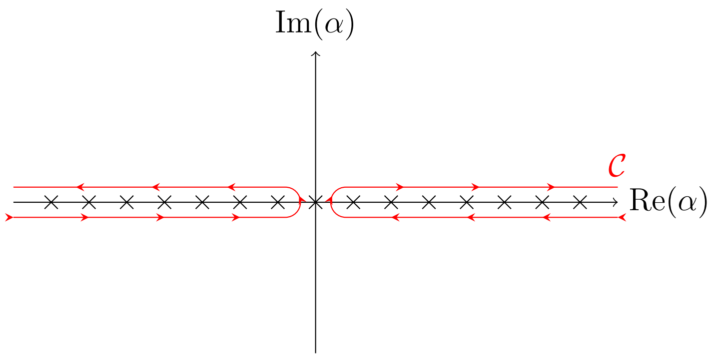

Figure 2: We deform the α integration contour to wrap the poles lying along the real axis.

weight labels in terms of μ and s we write this as

$$\begin{aligned}\log Z_{\Delta,\mathsf{s}} &= \sum_{\pm}\left(-\frac{1}{4\pi^2}\mathrm{Li}_3\left(e^{\mp 2\pi\mu}\right) \mp \frac{\mu}{2\pi}\mathrm{Li}_2\left(e^{\mp 2\pi\mu}\right) - \frac{\mu^2+\mathsf{s}^2}{2}\mathrm{Li}_1\left(e^{\mp 2\pi\mu}\right)\right)\\ &= 2\log Z_{\mathrm{scalar}} - \frac{\mathsf{s}^2}{2}\sum_{\pm}\mathrm{Li}_1\left(e^{\mp 2\pi\mu}\right) \;,\end{aligned} \tag{51}$$

where

$$\mathrm{Li}_p(z) = \sum_{n=1}^{\infty}\frac{z^n}{n^p} \tag{52}$$

is the polylogarithm. As mentioned above, $\log Z_{\mathrm{scalar}}$ is the one-loop determinant of a massive scalar field on S^3 with a mass set by $\ell_{\mathrm{dS}}^2 m^2 = \Delta(2-\Delta) = 1+\mu^2$. Up to an overall phase, unfixed by the quasinormal mode method, (51) matches the path integral of a massive spin-s field on S^3 via DHS in and the results reported in [14].

6 dS3 gravity and Chern-Simons theory

In this section we will review the relation between Chern-Simons theory and three-dimensional general relativity with a positive cosmological constant, i.e., dS_3 gravity [24]. Our presentation follows the work of [13, 14], which discusses the tree-level and loop relation between the two theories.

In the first half of this section we review geometrical properties of dS_3, and then classical (tree-level) aspects of the theories at hand. The second half is devoted to quantum aspects of dS_3. Our aim is to capture perturbative corrections to all orders in G_N via the Chern-Simons formulation. An important and novel portion of our analysis is to alter known methods to quantize Chern-Simons theory such that we meet the basic features that give Chern-Simons theory a gravitational interpretation. These alterations are to incorporate a non-trivial background connection and a complex level in the path integral. This is done in section 6.3, where we show how to adapt the derivation of the exact Chern-Simons path integral on S^3 via two different methods commonly used in the literature: Abelianisation and supersymmetric localization. These modifications are in perfect agreement with perturbative results in the metric formulation of the gravitational theory.

6.1 A primer on dS3 spacetime

Three-dimensional Lorentzian de Sitter space can be realised as the hypersurface in $\mathbb{R}^{1,3}$ (what we will call embedding space) given by

$$\eta_{AB} X^A X^B = \ell^2 \ , \qquad \eta = \mathrm{diag}(-1, 1, 1, 1) \ , \tag{53}$$

where $A, B \in \{0, 1, 2, 3\}$ and ℓ is the dS_3 radius. Embedding space makes it manifest that the isometry group of dS_3 is $SO(1,3)$ which is generated by Killing vectors preserving this hypersurface

$$L_{AB} = X_A \frac{\partial}{\partial X^B} - X_B \frac{\partial}{\partial X^A} \ . \tag{54}$$

Different parametrizations of (53) give different coordinate patches of global de Sitter. A particular coordinate patch of interest to us in this paper is the coordinate patch available to an observer moving along a timelike geodesic, called the static patch. Due to the accelerated expansion of the spacetime, individual observers lose causal contact with increasing portions of space which become hidden behind a causal horizon. Thus the static patch covers a finite causal

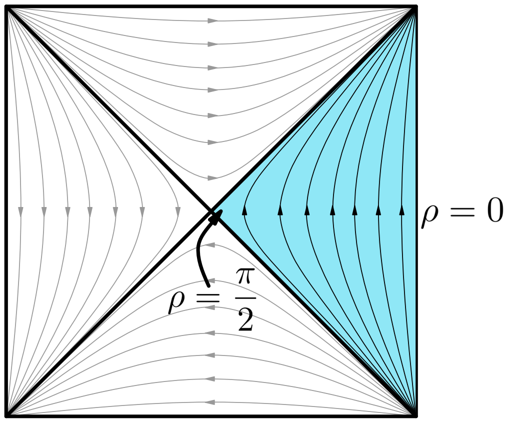

Figure 3: The Penrose diagram of dS_3. The blue region is the static patch covered by the coordinates (55). The observer defining the patch lies at $\rho = 0$. Their causal horizon lies at $\rho = \frac{\pi}{2}$. Also depicted are the flow lines of D which are time-like in this patch.

diamond, depicted as the blue region of the Penrose diagram found in figure 3. The parametrization for this static patch is given by

$$\begin{aligned} X^0 =& \ell \cos(\rho) \sinh(t/\ell) \ , \\ X^1 =& \ell \cos(\rho) \cosh(t/\ell) \ , \\ X^2 =& \ell \sin(\rho) \cos(\varphi) \ , \\ X^3 =& \ell \sin(\rho) \sin(\varphi) \ , \end{aligned} \tag{55}$$

for which the metric takes the following form

$$ds^2 = \eta_{AB} \mathrm{d}X^A \mathrm{d}X^B = -\cos^2\rho\, \mathrm{d}t^2 + \ell^2 \mathrm{d}\rho^2 + \ell^2 \sin^2\rho\, \mathrm{d}\varphi^2 \ . \tag{56}$$

The coordinates range over $t \in (-\infty, \infty)$, $\rho \in [0, \pi/2)$, $\varphi \in [0, 2\pi)$ which covers the right-wedge ("north-pole") of the static patch. The point $\rho = 0$ corresponds to the worldline of the observer defining the static patch, while $\rho = \pi/2$ corresponds to this observer's causal horizon. The metric (56) has an obvious time-like Killing vector, $\zeta = \partial_t$. This Killing vector is in fact the same as the "dilatation" Killing vector, $D = L_{03}$, of $\mathfrak{so}(1,3)$. As depicted in figure 3, this Killing vector is not globally time-like, however.

Euclidean de Sitter can be defined through the Wick rotation $X^0 = -iX^0_E$, which at the level of the static patch coordinates can be achieved through $t = -i\ell\tau$. The defining equation (53) then defines a three-sphere and indeed the Lorentzian static patch metric rotates to

$$\frac{ds^2}{\ell^2} = \cos^2\rho\,\mathrm{d}\tau^2 + \mathrm{d}\rho^2 + \sin^2\rho\,\mathrm{d}\varphi^2\ , \tag{57}$$

which is the metric for S^3 in torus coordinates. Regularity at the horizon, $\rho = \pi/2$, requires the identification $\tau \sim \tau + 2\pi$, consistent with it being a coordinate for S^3. The isometry group of Euclidean de Sitter is easily seen to be $SO(4) \simeq SU(2) \times SU(2)/\mathbb{Z}_2$. The $SU(2)$'s are the left and right group actions acting on S^3 which itself is diffeomorphic to $SU(2)$. As such we will label these two groups by subscripts L and R.

6.2 Chern-Simons theory and dS3 gravity on tree-level

Now let us briefly review the Chern-Simons formulation of three-dimensional gravity [24]; see [13] for more details and complementary aspects. Much like the previous subsection, this portion is intended to lay out the necessary ingredients and to establish our notation.

As we noted in section 6.1, the splitting $\mathfrak{so}(4) \simeq \mathfrak{su}(2)_L \oplus \mathfrak{su}(2)_R$ of the isometry algebra of Euclidean dS_3 indicates that we will be interested in quantizing a pair of $SU(2)$ Chern-Simons theories

$$S = k_L S_{\mathrm{CS}}[A_L] + k_R S_{\mathrm{CS}}[A_R]\ , \tag{58}$$

where

$$S_{\mathrm{CS}}[A] = \frac{1}{4\pi}\,\mathrm{Tr}\int_M \left(A\wedge \mathrm{d}A + \frac{2}{3}A^3\right)\ , \tag{59}$$

and the trace is taken in the fundamental representation. The levels, $k_{L/R}$, will be non-integer and ultimately related to G_N^{-1}. Following [25], the correct framework for approaching this theory is through its complexification $\mathfrak{sl}(2,\mathbb{C})$ with $\mathfrak{su}(2)$ taken as a real form. As emphasized in that paper a decomposition of levels consistent with reality of the action and with Euclidean gravity with positive cosmological constant is given by[8]

$$k_L = \delta + is\ , \qquad k_R = \delta - is\ , \tag{60}$$

[8] Strictly speaking, $\mathfrak{sl}(2,\mathbb{C})$ Chern-Simons theory parameterized in this way describes Lorentzian gravity

where $\delta \in \mathbb{Z}$ and $s \in \mathbb{R}$. As further discussed in [25], quantum effects lead to a finite renormalization of the levels

$$k_L \to r_L = k_L + 2 \ , \qquad k_R \to r_R = k_R + 2 \ . \tag{61}$$

Importantly these are renormalized in the same way and can be regarded as a renormalization of δ to $\hat{\delta} = \delta + 2$. For the rest of this section we will work with the renormalized levels.

To see that indeed this can be related to a theory of gravity, we can decompose the connections as

$$A_L = i\left(\omega^a + \frac{1}{\ell}e^a\right)L_a \ , \qquad A_R = i\left(\omega^a - \frac{1}{\ell}e^a\right)\bar{L}_a \ , \tag{62}$$

where $\{L_a\}$ and $\{\bar{L}_a\}$ generate $\mathfrak{su}(2)_L$ and $\mathfrak{su}(2)_R$, respectively.[9] It is natural to interpret e^a as the dreibein and $\omega^a = \frac{1}{2}\varepsilon^{abc}\omega_{bc}$ is the (dual) spin-connection. Indeed the action (58) is equivalent to

$$iS = -I_{\text{EH}} - i\hat{\delta} I_{\text{GCS}} \ , \tag{64}$$

where I_{EH} is the Einstein-Hilbert action written in first-order (or Palatini),

$$I_{\text{EH}} = -\frac{s}{4\pi\ell}\int \varepsilon_{abc}e^a \wedge \left(R^{bc} - \frac{1}{3\ell^2}e^b \wedge e^c\right) \ . \tag{65}$$

Here $R^{ab} = \varepsilon^{ab}{}_c\left(\mathrm{d}\omega^c - \frac{1}{2}\varepsilon^c{}_{de}\omega^d \wedge \omega^e\right)$ is the Riemann two-form, and we have a positive cosmological constant, $\Lambda = \ell^{-2}$. This identifies the imaginary part of the levels with Newton's constant

$$s = \frac{\ell}{4G_N} \ , \tag{66}$$

which establishes that the semi-classical regime of this theory is the large s limit. The second part of this action, once restricted

with positive cosmological constant and with $\mathfrak{sl}(2,\mathbb{R})$ as its real form. We obtain the Euclidean theory from the Wick rotation: i.e. (supposing the negative sign of the metric is associated with e^3) $e^3 \to ie^3$, $L_3 \to iL_3$.

[9] With respect to this basis, we have

$$\mathrm{Tr}(L_a L_b) = \mathrm{Tr}\big(\bar{L}_a \bar{L}_b\big) = \frac{1}{2}\delta_{ab} \ . \tag{63}$$

to torsion-free spin connections, is the gravitational Chern-Simons action:

$$I_{\rm GCS} = \frac{1}{2\pi}\int {\rm Tr}\left(\omega\wedge d\omega + \frac{2}{3}\omega^3\right) + \frac{1}{2\pi\ell^2}\int {\rm Tr}\left(e\wedge T\right) \ , \tag{67}$$

where $T^a = {\rm d}e^a - \varepsilon^a{}_{bc}\omega^b\wedge e^c$ is the torsion two-form.

It is also simple to establish a relation at the level of the equations of motion. The classical equations of motion of the Chern-Simons theories (58) are

$${\rm d}A_L + A_L\wedge A_L = 0 \ , \qquad {\rm d}A_R + A_R\wedge A_R = 0 \ . \tag{68}$$

The sum and difference of these equations translate, in terms of e^a and ω^a, to the vacuum Einstein equation (with positive cosmological constant) and the vanishing of the torsion two-form:

$$R^{ab} = \frac{1}{\ell^2}e^a\wedge e^b \ , \qquad T^a = 0 \ . \tag{69}$$

These derivations establish a correspondence between classical solutions in the metric formulation of dS_3 gravity and classical solutions in the Chern-Simons theory.

Background configuration It will be important to make explicit how to cast Euclidean dS_3 space, i.e., the three-sphere, in the language of Chern-Simons theory. We start by constructing the appropriate flat connections on S^3, which we will coin (a_L, a_R). Given the metric (57), a convenient choice of dreibein is

$$e^1 = \ell{\rm d}\rho \ , \qquad e^2 = \ell\sin\rho\,{\rm d}\varphi \ , \qquad e^3 = \ell\cos\rho\,{\rm d}\tau \ , \tag{70}$$

with associated torsion-free spin connection

$$\omega^1 = 0 \ , \qquad \omega^2 = -\sin\rho\,{\rm d}\tau \ , \qquad \omega^3 = -\cos\rho\,{\rm d}\varphi \ . \tag{71}$$

From these expressions, we find

$$\begin{aligned} a_L &= iL_1{\rm d}\rho + i\left(\sin\rho\,L_2 - \cos\rho\,L_3\right)({\rm d}\varphi - {\rm d}\tau) \\ &= g_\rho^{-1}g_-^{-1}{\rm d}\left(g_-g_\rho\right) \ , \\ a_R &= -i\bar{L}_1{\rm d}\rho - i\left(\sin\rho\,\bar{L}_2 + \cos\rho\,\bar{L}_3\right)({\rm d}\varphi + {\rm d}\tau) \\ &= -{\rm d}\left(g_\rho g_+\right)g_+^{-1}g_\rho^{-1} \ , \end{aligned} \tag{72}$$

where we used (62). The second equality of each line above emphasizes that a_L and a_R are pure gauge with

$$g_\rho = e^{iL_1\rho} \ , \qquad g_\pm = e^{-iL_3(\tau\pm\varphi)} \ . \tag{73}$$

The connections (72) are locally flat, however they possess a point-like singularity. These are singularities for $\mathrm{d}\tau$ and $\mathrm{d}\varphi$ at $\rho = \pi/2$ and $\rho = 0$, respectively. These will be treated, as distributions, by

$$\mathrm{d}(\mathrm{d}\varphi) = \delta(\rho)\mathrm{d}\rho \wedge \mathrm{d}\varphi \ , \qquad \mathrm{d}(\mathrm{d}\tau) = -\delta(\rho - \pi/2)\mathrm{d}\rho \wedge \mathrm{d}\tau \ . \tag{74}$$

It is simple to extract the holonomies of a_L and a_R, which are important to record for later use. For any cycle γ wrapping the singular points of the connections, the connections possess holonomies

$$\mathcal{P}\exp \oint_\gamma a_L = g_\rho^{-1} e^{i2\pi L_3 \mathsf{h}_L} g_\rho \ , \qquad \mathcal{P}\exp \oint_\gamma a_R = g_\rho e^{i2\pi \bar{L}_3 \mathsf{h}_R} g_\rho^{-1} \ . \tag{75}$$

Requiring that the above group elements' action on $S^3 \simeq SU(2)$ itself is single-valued implies that $\mathsf{h}_L, \mathsf{h}_R \in \mathbb{Z}$ with either both even or both odd.[10] In particular, for cycles wrapping the causal horizon at $\rho = \frac{\pi}{2}$, we have

$$\mathsf{h}_L = 1 \ , \qquad \mathsf{h}_R = -1 \ . \tag{76}$$

Finally, we report on the value of the on-shell action for this background. A short calculation, which uses (74), shows that they have non-trivial action

$$r_L S_{\mathrm{CS}}[a_L] = -\pi r_L = -\pi\hat{\delta} - i\pi s \ , \ r_R S_{\mathrm{CS}}[a_R] = \pi r_R = \pi\hat{\delta} - i\pi s \ , \tag{77}$$

and thus

$$\begin{aligned} iS|_{\mathrm{tree-level}} &= i r_L S_{\mathrm{CS}}[a_L] + i r_R S_{\mathrm{CS}}[a_R] \\ &= \frac{\pi \ell}{2G_N} \ , \end{aligned} \tag{78}$$

where we used (66). This is the correct on-shell action for dS_3 [13]. Note that the gravitational Chern-Simons term of S^3 vanishes identically.

[10] Namely, this geometric action is in the fundamental representation. In that case $e^{i2\pi L_3 \mathsf{h}_{L/R}}$ is obviously the identity if $\mathsf{h}_{L/R}$ is even. If h_L and h_R are both odd this yields the group element $(-1,-1) \in SU(2)_L \times SU(2)_R$ which is also the identity inside the $\mathbb{Z}_2$ quotient.

6.3 SU(2) Chern-Simons theory: the partition function

We now turn to quantum aspects of dS_3 gravity. Our aim is to perform the gravitational path-integral about a fixed background S^3 saddle. This will be done in the Chern-Simons formulation of the theory which we introduced in section 6.2.

It is well-known that many observables in Chern-Simons theory can be evaluated exactly, i.e., to all orders in perturbation theory and also including non-perturbative effects. However, for our gravitational purpose, some caution is needed since these results are not always applicable due to the subtle relation between Chern-Simons and gravity. Here we will address these subtleties at the level of evaluating the path integral on S^3. In a nutshell, we will re-derive $Z_k[S^3]$ for $SU(2)$ Chern-Simons theory with level k, while allowing the level to be complex and also allowing non-trivial background connections. These are two key features that are persistent in the relation among the two theories, as we reviewed in the previous subsection.

Let us therefore begin by reviewing some basic facts and definitions. The Chern-Simons partition function over a three-manifold, M, is the path-integral

$$Z_k[M] = \int \frac{\mathcal{D}A}{\mathcal{V}}\, e^{ikS_{\mathrm{CS}}[A]} \tag{79}$$

over the action (59). Here $A = A^a L_a$ is to be regarded as a connection one-form of a principal $SU(2)$ bundle over M, where $\{L_a\}$ generates the $\mathfrak{su}(2)$ Lie algebra.[11] In the measure we indicate, schematically, a division by the gauge group as $1/\mathcal{V}$.

There are three remarks that will be important in what follows. First, the action (59) is clearly topological and the quantum theory itself is almost topological: its sole geometric input is a choice of framing which arises from regularizing the phase of Z_k. While there is no "rule" for establishing the framing, partition functions differing by choices of framing are related by well-established phases [11].

[11] Note that given the form of (59), we are working with the convention that A is anti-Hermitian in the fundamental representation, i.e. $A^a_\mu \in i\mathbb{R}$. We will use this convention consistently throughout.

In this paper we will be careful to work with a fixed convention for the phase of Z_k.[12]

Second, our evaluation of (79) will cover complex values of the level k. In particular, our derivations will hold for a decomposition as in done in (60)-(61).

Third, we will incorporate a flat background connection to the path integral (79). To that end we will write

$$A = a + B \ . \tag{80}$$

Here a is a flat background connection on M—for most of our purposes $M = S^3$. It is important to emphasize at this point that, unlike what is typical for Chern-Simons theory quantized on S^3, we will not take the trivial background $a = 0$: such a background leads to a degenerate metric which is an unnatural saddle for a theory of gravity. Instead we want connections corresponding to a round S^3 metric, i.e., they will be (72), with holonomies (75)-(76), for each copy of the $SU(2)$ theory. The field B captures the quantum fluctuations that we will integrate over in the path integral shortly afterward.

Adapting exact results We now turn to the tools we will use for evaluating $Z_k[S^3]$. There are several ways to obtain $Z_k[S^3]$, and we do not attempt to describe them all. We selected methods for which the choice of background connections and background topology (and later, expectation values for Wilson loops) are tractable in the path integral of $SU(2)_k$ Chern-Simons theory. The two methods we will discuss in detail are:

Abelianisation. The process of Abelianisation was developed in [18, 26]. In a nutshell, it demonstrates how the non-Abelian Chern-Simons path integral can be reduced to a two-dimensional Abelian theory, under suitable conditions present on the manifold M.

Supersymmetric localization. As a complementary method, we will show how one obtains $Z_k[S^3]$ via supersymmetric localization tech-

[12] Which is ultimately related to two-units away from so-called "canonical framing".

niques [27] (see also [20]). The biggest penalty here is the introduction of fermions in the path integral. Still, the outcome is robust and completely agrees with Abelianisation.

Both methods will be capable of successfully accommodating the features necessary for dS_3 gravity, and we stress that they report the same result (up to a trivial normalization). This subsection will summarize the main steps of both methods, highlighting in particular the features that need to be altered to accommodate gravity.

6.3.1 N=2 supersymmetric localization

We now describe an alternative route to the exact calculation of the Chern-Simons partition through localization techniques. We will focus particularly on $\mathcal{N} = 2$ supersymmetric localization [27]. One benefit of this approach is that much of the basic machinery has been established with a non-trivial background connection, a, in mind allowing a fairly straightforward incorporation of $a \neq 0$. However: the situations with non-trivial background connections have historically arisen on manifolds with interesting topology (e.g. Lens spaces) and many of the explicit results for S^3 have been established with $a = 0$. Below we collect and synthesize these results in a way that is useful for dS_3 gravity.

Before jumping in, let us also make the following brief comments. Supersymmetry in the context of de Sitter is a contentious subject, with much of the difficulty arising from realizing unitary representations of the supersymmetry algebra in Lorentzian signature. Here a somewhat agnostic stance is taken on this topic: by working directly in Euclidean signature, we are ultimately discussing $SU(2)_k$ Chern-Simons theory on S^3 whose $\mathcal{N} = 2$ supersymmetric extension is well-established. We use the existence of this symmetry to our advantage to localize the path integral all while verifying that the extension to $\mathcal{N} = 2$ does not alter essential features of the original partition function.

Let us set the stage and collect the necessary background. Much of what follows mirrors the friendly review [20]. The vector multiplet of three dimensional $\mathcal{N} = 2$ gauge theory is given by fields

$$\{A_\mu,\ \boldsymbol{\sigma},\ \mathfrak{D},\ \lambda,\ \bar{\lambda}\}\ , \tag{81}$$

where A is a $\mathfrak{g} = \mathfrak{su}(2)$ connection, $\boldsymbol{\sigma}$, $\mathfrak{D}$ are scalars, and λ, $\bar{\lambda}$ are Dirac spinors. Note the same field content as in table 1 and (5) but without charge, color and the axion [2, 6].

All fields are $\mathfrak{g}$-valued and by convention we will take them all to be anti-Hermitian,[13] with supersymmetry variations parameterized by two Grassmann variables $\bar{\epsilon}$ and ϵ as specified in [20]. The supersymmetric Chern-Simons action is

$$S_{\mathrm{SCS}} = \frac{1}{4\pi}\int \mathrm{Tr}\left(A \wedge \mathrm{d}A + \frac{2}{3}A^3\right) - \frac{1}{4\pi}\int d^3x\sqrt{g}\,\mathrm{Tr}\left(\bar{\lambda}\lambda - 2\mathfrak{D}\boldsymbol{\sigma}\right)\ , \tag{82}$$

and enters the path integral multiplied by the level k

$$Z_k^{\mathrm{SCS}}[S^3] = \int \frac{\mathcal{D}A}{\mathcal{V}_G}\mathcal{D}\bar{\lambda}\mathcal{D}\lambda\mathcal{D}\mathfrak{D}\mathcal{D}\boldsymbol{\sigma}\, e^{ikS_{\mathrm{SCS}}}\ . \tag{83}$$

To make subsequent notation less cumbersome, we will drop the "$[S^3]$" above with it understood that we are always working on the three sphere. Note that on a formal level, as far as the function dependence on k is concerned, the addition of the auxiliary fields in the multiplet does not alter Z_k^{SCS} with respect to the non-supersymmetric path integral, Z_k [20].

The deformation that allows us to localize the path integral Z_k^{SCS} is the super Yang-Mills action

$$\begin{aligned} S_{\mathrm{SYM}} = &-\int \mathrm{Tr}\left(\frac{1}{2}F \wedge \star F + D\boldsymbol{\sigma} \wedge \star D\boldsymbol{\sigma}\right) \\ &-\int d^3x\sqrt{g}\,\mathrm{Tr}\left(\frac{1}{2}\left(\mathfrak{D} + \boldsymbol{\sigma}\right)^2 + \frac{i}{2}\bar{\lambda}\gamma^\mu D_\mu\lambda - \frac{1}{2}\bar{\lambda}[\boldsymbol{\sigma}, \lambda] - \frac{1}{4}\bar{\lambda}\lambda\right)\ , \end{aligned} \tag{84}$$

where D_μ is the gauge covariant derivative and γ_μ can be taken to be the Pauli-matrices acting on spinor indices. S_{SYM} is itself a

[13] In comparison to the notation of [20], a field here is related to a field there by $\Phi_{\mathrm{here}} = i\Phi_{\mathrm{there}}$.

super-derivative and therefore Q-exact. Adding this to the path integral with coefficient $\mathtt{t}$, i.e.,

$$Z_k^{\text{SCS+SYM}}(\mathtt{t}) = \int \frac{\mathcal{D}A}{\mathcal{V}_G} \mathcal{D}\bar{\lambda}\mathcal{D}\lambda\mathcal{D}D\mathcal{D}\boldsymbol{\sigma}\, e^{ikS_{\text{SCS}} - \mathtt{t}S_{\text{SYM}}} \ , \tag{85}$$

is then innocuous: $Z_k^{\text{SCS+SYM}}(\mathtt{t}) = Z_k^{\text{SCS}}$ for any $\mathtt{t}$, including in the limit $\mathtt{t} \to \infty$ where the path-integral localizes on the saddle of S_{SYM}.

Localization locus

In the $\mathtt{t} \to \infty$ limit, the path integral localizes on the following equations of motion

$$F = 0 \ , \qquad D\boldsymbol{\sigma} = \mathrm{d}\boldsymbol{\sigma} + [A, \boldsymbol{\sigma}] = 0 \ , \qquad \mathfrak{D} + \boldsymbol{\sigma} = 0 \ . \tag{86}$$

We expand the solutions around a flat connection $a = g^{-1}\mathrm{d}g$, for some group element g. Again, g may not be single-valued and a may possess a holonomy

$$\mathcal{P}\exp\left(\oint_\gamma a\right) = \exp(2\pi\mathfrak{m}) \ , \tag{87}$$

for some curve γ. The other fields that have saddle solutions to (86) are given by

$$\boldsymbol{\sigma}_0^{(g)} = g^{-1}\boldsymbol{\sigma}_0 g \ , \qquad \mathfrak{D}_0 = -\boldsymbol{\sigma}_0^{(g)} \ , \qquad \lambda_0 = 0 \ , \qquad \bar{\lambda}_0 = 0 \ , \tag{88}$$

for $\boldsymbol{\sigma}_0$ a constant element of $\mathfrak{g}$. We require $\boldsymbol{\sigma}_0^{(g)}$ to be single-valued and so the constant element defining the saddle must obey

$$[\mathfrak{m}, \boldsymbol{\sigma}_0] = 0 \ . \tag{89}$$

With this we can take $\boldsymbol{\sigma}_0$ to be in a Cartan subalgebra containing $\mathfrak{m}$. We will scale fluctuations as

$$\begin{aligned} &A = a + \frac{1}{\sqrt{\mathtt{t}}}B \ , \qquad \boldsymbol{\sigma} = \boldsymbol{\sigma}_0^{(g)} + \frac{1}{\sqrt{\mathtt{t}}}\hat{\sigma} \ , \qquad \mathfrak{D} = -\boldsymbol{\sigma}_0^{(g)} + \frac{1}{\sqrt{\mathtt{t}}}\hat{\mathfrak{D}} \ , \\ &\lambda = \frac{1}{\sqrt{\mathtt{t}}}\hat{\lambda} \ , \qquad \bar{\lambda} = \frac{1}{\sqrt{\mathtt{t}}}\hat{\bar{\lambda}} \ , \end{aligned} \tag{90}$$

and perturb the action (82) around the saddle (88) as $\mathsf{t} \to \infty$. The leading contribution to S_{SCS} is

$$\lim_{\mathsf{t}\to\infty} S_{\text{SCS}} = S_{\text{CS}}[a] - \frac{\text{Vol}(S^3)}{2\pi}\text{Tr}\boldsymbol{\sigma}_0^2 \,. \tag{91}$$

Meanwhile the leading contribution to $\mathsf{t}\, S_{\text{SYM}}$ is

$$\begin{aligned}\mathsf{t}\, S_{\text{SYM}} = &- \int \text{Tr}\left(\frac{1}{2}\text{d}_a B \wedge \star \text{d}_a B + (\text{d}_a\hat{\sigma} + [B, \boldsymbol{\sigma}_0^{(g)}]) \wedge \star(\text{d}_a\hat{\sigma} + [B, \boldsymbol{\sigma}_0^{(g)}])\right) \\ &- \int d^3x\sqrt{g}\text{Tr}\left(\frac{1}{2}\left(\hat{\mathfrak{D}} + \hat{\sigma}\right)^2 + \frac{i}{2}\hat{\bar{\lambda}}\gamma^\mu D_\mu^{(a)}\hat{\lambda} - \frac{1}{2}\hat{\bar{\lambda}}[\boldsymbol{\sigma}_0^{(g)}, \hat{\lambda}] - \frac{1}{4}\hat{\bar{\lambda}}\hat{\lambda}\right) + \ldots\end{aligned} \tag{92}$$

where d_a is the background exterior derivative, and $D_\mu^{(a)}$ is the spinor covariant derivative with fixed connection, a. This action can be made Gaussian under a suitable gauge fixing and then path integrated in standard fashion. We briefly highlight the main points of that procedure below, but many details can be found [20] and references therein.

Gauge choice

We will choose the gauge[14]

$$\mathcal{G}_a[B] = \text{d}_a^\dagger B \equiv - \star \text{d}_a \star B = 0 \,, \tag{93}$$

whose Fadeev-Popov determinant, $\Delta_a[B]$, can be enacted through adding ghosts $\bar{c}, c$:

$$\begin{aligned}Z_k^{\text{SCS+SYM}} = &\, e^{ikS_{\text{CS}}[a]} \int \text{d}\boldsymbol{\sigma}_0\, e^{-i\frac{k}{2\pi}\text{vol}(M_3)\text{Tr}\boldsymbol{\sigma}_0^2} \\ &\times \int \frac{\mathcal{D}B}{\mathcal{V}}\mathcal{D}\hat{\bar{\lambda}}\mathcal{D}\hat{\lambda}\mathcal{D}\hat{\mathfrak{D}}\mathcal{D}\hat{\sigma}\mathcal{D}\bar{c}\mathcal{D}c\,\delta[\text{d}_a^\dagger B]\, e^{-\mathsf{t}S_{\text{SYM}} - S_{\text{ghost}}} \,,\end{aligned} \tag{94}$$

with action

$$S_{\text{ghost}} = \int \text{Tr}\left(\bar{c} \wedge \star \text{d}_a^\dagger \text{d}_{a+\mathsf{t}^{-1/2}B}\, c\right) = \int d^3x\sqrt{g}\,\text{Tr}\left(\bar{c} \wedge \star\Delta_a^0\, c\right) + O(\mathsf{t}^{-1/2}) \,, \tag{95}$$

[14] This gauge fixing is only consistent when a is a flat-connection, implying that $\text{d}_a^2 = 0$ defines an equivariant cohomology.

where $\Delta_a^0 = \mathrm{d}_a^\dagger \mathrm{d}_a$ is the a-deformed Laplacian acting on $\mathfrak{g}$-valued zero-forms.[15] The ghost determinants simply cancel the determinants from $\hat{\mathfrak{D}}$ and $\hat{\sigma}$ (as well as a Jacobian from $\delta[\mathrm{d}_a^\dagger B]$) and so we arrive at the promised Gaussian path-integral:

$$Z_k^{\mathrm{SCS+SYM}} = e^{ikS_{\mathrm{CS}}[a]} \int \mathrm{d}\boldsymbol{\sigma}_0 \, e^{-i\frac{k}{2\pi}\mathrm{vol}M_3 \mathrm{Tr}\boldsymbol{\sigma}_0^2} \, Z_{\mathrm{Gauss}}[\boldsymbol{\sigma}_0] \ , \tag{96}$$

with

$$Z_{\mathrm{Gauss}}[\boldsymbol{\sigma}_0] := \int [\mathcal{D}B]_{\mathrm{ker}\mathrm{d}_a^\dagger} \mathcal{D}\hat{\bar{\lambda}} \mathcal{D}\hat{\lambda} \, e^{\frac{1}{2}\int \mathrm{Tr}(\mathrm{d}_a B)^2 + \int \mathrm{Tr}[B, \boldsymbol{\sigma}_0^{(g)}]^2 - \int \mathrm{Tr}\left(\frac{i}{2}\hat{\bar{\lambda}}\gamma_\mu D_\mu^{(a)}\hat{\lambda} - \frac{1}{2}\hat{\bar{\lambda}}[\boldsymbol{\sigma}_0^{(g)}, \hat{\lambda}] - \frac{1}{4}\hat{\bar{\lambda}}\hat{\lambda}\right)} \, . \tag{97}$$

One loop determinants

The remaining task is now to compute the one loop determinants from integrating out $\{B, \hat{\bar{\lambda}}, \hat{\lambda}\}$. The first step is to "canonicalize" the kinetic terms by redefining the fluctuating fields $\{B, \hat{\bar{\lambda}}, \hat{\lambda}\} \to \{\tilde{B}, \tilde{\bar{\lambda}}, \tilde{\lambda}\}$ via

$$\Phi = g^{-1}\tilde{\Phi} g \ , \qquad \Phi \in \{B, \hat{\bar{\lambda}}, \hat{\lambda}\} \ . \tag{98}$$

As a result the one loop integration becomes ostensibly simpler

$$Z_{\mathrm{Gauss}}[\boldsymbol{\sigma}_0] = \int [\mathcal{D}\tilde{B}]_{\mathrm{ker}d^\dagger} [\mathcal{D}\tilde{\bar{\lambda}} \mathcal{D}\tilde{\lambda}] e^{\frac{1}{2}\int \mathrm{Tr}\left(\mathrm{d}\tilde{B}\right)^2 + \int \mathrm{Tr}[\tilde{B}, \boldsymbol{\sigma}_0]^2 - \int \mathrm{Tr}\left(\frac{i}{2}\tilde{\bar{\lambda}}\gamma^\mu \nabla_\mu \tilde{\lambda} - \frac{1}{2}\tilde{\bar{\lambda}}[\boldsymbol{\sigma}_0, \tilde{\lambda}] - \frac{1}{4}\tilde{\bar{\lambda}}\tilde{\lambda}\right)} \ , \tag{99}$$

however, as we saw earlier, this is at the cost of twisting the fields along the curve γ:

$$\tilde{\Phi}_f = \exp(-2\pi\tilde{\mathfrak{m}})\tilde{\Phi}_i \exp(2\pi\tilde{\mathfrak{m}}) \ . \tag{100}$$

In terms of a root space decomposition $\tilde{\Phi} = \tilde{\Phi}^{(i)} T_i + \tilde{\Phi}^{(\alpha)} T_\alpha$, then (100) reads

$$\tilde{\Phi}_f^{(i)} = \tilde{\Phi}_i^{(i)} \ , \qquad \tilde{\Phi}_i^{(\alpha)} = \exp(-2\pi\alpha \cdot \tilde{\mathfrak{m}})\tilde{\Phi}_i^{(\alpha)} \ , \tag{101}$$

[15] It is tacit in (94) that the zero modes of $\bar{c}, c$ under Δ_a^0 are not to be integrated over.

where T_i is a basis of the Cartan subalgebra containing $\boldsymbol{\sigma}_0$ and $\mathfrak{m}$, and T_α are a basis of the α root space; $\{\tilde{\Phi}^{(i)}, \tilde{\Phi}^{(\alpha)}\}$ are honest fields and not elements of $\mathfrak{g}$. The one loop determinants of the $\mathcal{N} = 2$ vector multiplet with twisted boundary conditions, (101), turns out to be

$$Z_{\text{Gauss}}[\boldsymbol{\sigma}_0] = \prod_{\alpha>0} \frac{\sin\left(i\alpha \cdot (\boldsymbol{\sigma}_0 - 2\pi\tilde{\mathfrak{m}})\right)}{\pi^2} \ . \tag{102}$$

Again, we have written Z_{Gauss} rather generally, but for the purposes of this paper, we can let $\tilde{\mathfrak{m}} = i\mathsf{h}\, L_3$ with $\mathsf{h} \in \mathbb{Z}$ in which case it reduces to the usual expression for the Ray-Singer torsion in terms of $\boldsymbol{\sigma}_0 = -i\sigma L_3$, i.e. $Z_{\text{Gauss}} = \sin^2(\pi\sigma)/\pi^2$. The phase of Z_{Gauss} again is responsible for the renormalization $k \to r = k + 2$ as explained in [20]. Gathering these results and fixing the normalization, we find

$$Z_k^{\text{SCS}} = e^{irS_{\text{CS}}[a]} \int_{\mathbb{R}} \mathrm{d}\sigma\, \sin^2(\pi(\sigma + \mathsf{h}))\, e^{i\frac{\pi}{2}r\sigma^2} \ , \tag{103}$$

where again the integration contour over σ should be deformed depending on the phase of r.

7 Outlook

Lattice methods have been developed for CS theory canonical quantization [28, 29]. These make possible developing numerical calculations.

The generation question is likely to solved by an additional symmetry group or excitation interaction.

The supersymmetric Chern-Simons actions, (5) and (82) (with matter) provide a new unified theory of matter and quantum gravity. The present discussion is precursory. Details of this framework have to be studied consistently. A unique single action has an element of a theory of "everything".

References

[1] Risto Raitio, A Model of Lepton and Quark Structure. Physica Scripta, 22, 197 (1980). PS 22,197. viXra:1903.0224
The core of this model was conceived in November 1974 at SLAC. I proposed that the c-quark would be an excitation of the u-quark, both composites of three 'subquarks'. The idea was opposed by the community and was therefore not written down until five years later.

[2] Risto Raitio, Supersymmetric preons and the standard model, Nuclear Physics B931, 283–290 (2018). DOI: 10.1016/j.nuclphysb.2018.04.021 arXiv:1805.03013

[3] J. Wess and B. Zumino, Supergauge transformations in four dimensions, Nucl. Phys. B 70, 39 (1974). DOI: 10.1016/0550-3213(74)90355-1

[4] Roberto D. Peccei and Helen R. Quinn, CP Conservation in the Presence of Pseudoparticles, Phys. Rev. Lett. 38 (25) 1440–1443 (1977).

[5] Risto Raitio, The fate of supersymmetry in quantum field theories, Journal of High Energy Physics, Gravitation and Cosmology, Vol.10, No.2 (2024). DOI: 10.4236/jhepgc.2024.102038. arXiv:2307.13017

[6] Risto Raitio, A Chern-Simons model for baryon asymmetry, Nuclear Physics B Volume 990, May 2023, 116174 . DOI: 10.1016/j.nuclphysb.2023.116174. arXiv:2301.10452

[7] H. Belich, O. M. Del Cima, M. M. Ferreira Jr. and J. A. Helayel-Neto, Electron-Electron Bound States in Maxwell-Chern-Simons-Proca QED_3, Eur. Phys. J. B 32, 145–155 (2003). arXiv:hep-th/0212285

[8] Ya. I. Kogan, JETP Lett. 49, 225 (1989).

[9] M. I. Dobroliubov, D. Eliezer, I. I. Kogan, G.W. Semenoff and R.J. Szabo, Mod. Phys. Lett. A, 8, 2177 (1993).

[10] Robert Bourne, Alejandra Castro, and Jackson R. Fliss, Spinning up the spool: Massive spinning fields in 3d quantum gravity. arXiv:2407.09608

[11] E. Witten, (2+1)-Dimensional Gravity as an Exactly Soluble System, Nucl. Phys. B 311 (1988) 46.

[12] Collier, Scott and Eberhardt, Lorenz and Zhang, Mengyang, Solving 3d gravity with Virasoro TQFT, SciPost Phys., 15, 4 (2023). DOI: 10.21468/SciPostPhys.15.4.151. arXiv:2304.13650

[13] Castro, Alejandra and Lashkari, Nima and Maloney, Alexander, A de Sitter Farey Tail, Phys. Rev. D 83, 124027 (2011). DOI: 10.1103/PhysRevD.83.124027. arXiv:1103.4620

[14] Anninos, Dionysios and Denef, Frederik and Law, Y. T. Albert and Sun, Zimo, Quantum de Sitter horizon entropy from quasicanonical bulk, edge, sphere and topological string partition functions, JHEP 01, 088 (2022). DOI: 10.1007/JHEP01(2022)088. arXiv:2009.12464

[15] Ooguri, Hirosi and Vafa, Cumrun, Knot invariants and topological strings, Nucl. Phys. B 577, 419-438 (2000). DOI: 10.1016/S0550-3213(00)00118-8. arXiv:hep-th/9912123

[16] A. Castro, I. Coman, J. R. Fliss, and C. Zukowski, Keeping matter in the loop in dS3 quantum gravity, JHEP 07 (2023) 120. arXiv:2302.12281

[17] He, Miao, One-loop partition functions in $T\overline{T}$-deformed AdS_3, JHEP 05, 067 (2024). DOI: 10.1007/JHEP05(2024)067. arXiv:2401.09879

[18] Blau, Matthias and Thompson, George, Chern-Simons Theory on Seifert 3-Manifolds, JHEP 09, 033 (2013). DOI: 10.1007/JHEP09(2013)033. arXiv:1306.3381

[19] Alejandra Castro, Ioana Coman, Jackson R. Fliss, and Claire Zukowski, Coupling Fields to 3D Quantum Gravity via Chern-Simons Theory, Phys. Rev. Lett. 131, 171602 (2023). DOI: 10.1103/PhysRevLett.131.171602. arXiv:2304.02668

[20] Marcos Mariño, Lectures on localization and matrix models in supersymmetric Chern–Simons–matter theories, J. Phys. A: Math. Theor. 44 463001 (2011). DOI: 10.1088/1751-8113/44/46/463001. arXiv:1104.0783

[21] Zinoviev, Yu. M., On massive high spin particles in (A)dS. arXiv:hep-th/0108192

[22] Denef, Frederik and Hartnoll, Sean A. and Sachdev, Subir, Black hole determinants and quasinormal modes, Class. Quant. Grav. 27,125001 (2010). DOI: 10.1088/0264-9381/27/12/125001. arXiv:0908.2657

[23] Penedones, Joao and Salehi Vaziri, Kamran and Sun, Zimo, Hilbert space of Quantum Field Theory in de Sitter spacetime (2023). arXiv:2301.04146

[24] Witten, Edward, Quantum Field Theory and the Jones Polynomial, Commun. Math. Phys. 121, 351-399 (1989). DOI: 10.1007/BF01217730.

[25] Witten, Edward, Quantization of Chern-Simons Gauge Theory with Complex Gauge Group, Commun. Math. Phys., 137, 29-66 (1991). DOI: 10.1007/BF02099116

[26] Blau, Matthias and Thompson, George, Chern-Simons theory on S1-bundles: Abelianisation and q-deformed Yang-Mills theory, JHEP 05, 003 (2006). DOI: 10.1088/1126-6708/2006/05/003. arXiv:hep-th/0601068

[27] Kapustin, Anton and Willett, Brian and Yaakov, Itamar, Exact Results for Wilson Loops in Superconformal Chern-Simons Theories with Matter, JHEP 03, 089 (2010). DOI: 10.1007/JHEP03(2010)089. arXiv:0909.4559

[28] Theodore Jacobson and Tin Sulejmanpasic, Canonical quantization of lattice Chern-Simons theory. arXiv:2401.09597

[29] Bingnan Zhang, Abelian Chern-Simons gauge theory on the lattice, Phys. Rev. D 105, 014507 (2022). arXiv:2109.13411